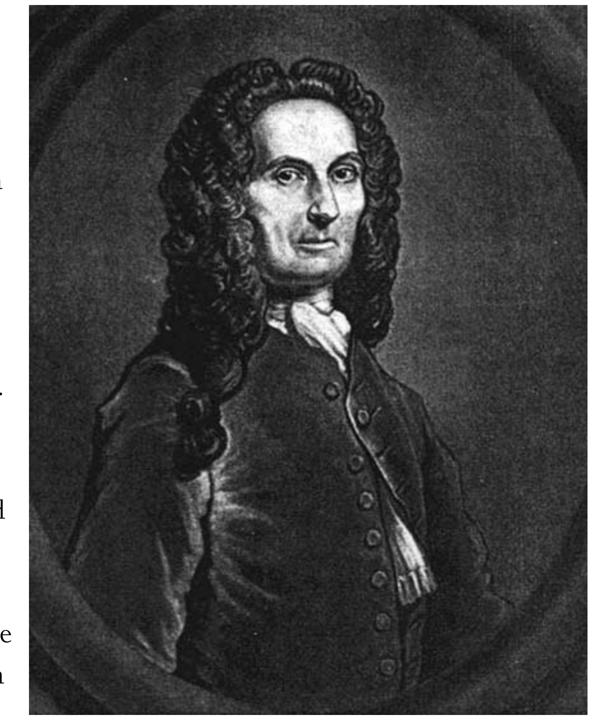
17 Complex Numbers



Abraham de Moivre

Abraham de Moivre was born in Vitry-le-François in France on 26 May 1667. It was not until his late teenage years that de Moivre had any formal mathematics training. In 1685 religious persecution of Protestants became very serious in France and de Moivre, as a practising Protestant, was imprisoned for his religious beliefs. The length of time for which he was imprisoned is unclear, but by 1688 he had moved to England and was a private tutor of mathematics, and was also teaching in the coffee houses of London. In the last decade of the 15th century he met Newton and his first mathematics paper arose from his study of fluxions in Newton's *Principia*. This first paper was accepted by the Royal Society in 1695 and in 1697 de Moivre was elected as a Fellow of the Royal Society. He researched mortality statistics and probability and during the first decade of the 16th century he published his theory of probability. In 1710 he was asked to evaluate the claims of Newton and Leibniz to be the discoverers of calculus. This was a major and important undertaking at the time and it is interesting that it was given to de Moivre despite the fact he had found it impossible to gain a university post in England. In many ways de Moivre is best known for his work with the formula $(\cos x + i \sin x)^n$. The theorem that comes from this bears his name and will be introduced in this chapter.

De Moivre was also famed for predicting the day of his own death. He noted that each night he was sleeping 15 minutes longer and by treating this as an arithmetic progression and summing it, he calculated that he would die on the day that he slept for 24 hours. This was 27 November 1754 and he was right!

17.1 Imaginary numbers

Up until now we have worked with any number k that belongs to the real numbers and has the property $k^2 \geq 0$. Hence we have not been able to find $\sqrt{\text{negative number}}$ and have not been able to solve equations such as $x^2 = -1$. In this chapter we begin by defining a new set of numbers called imaginary numbers and state that $i = \sqrt{-1}$.

An **imaginary number** is any number of the form

$$\begin{aligned}\sqrt{-n^2} &= \sqrt{n^2 \times -1}\\ &= \sqrt{n^2} \times \sqrt{-1}\\ &= ni\end{aligned}$$

Adding and subtracting imaginary numbers

Imaginary numbers are added in the usual way and hence $3i + 7i = 10i$.

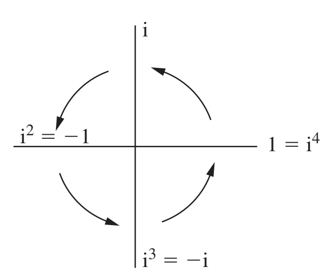
They are also subtracted in the usual way and hence $3i - 7i = -4i$.

Multiplying imaginary numbers

When we multiply two imaginary numbers we need to consider the fact that powers of i can be simplified as follows:

$$\begin{aligned}i^2 &= i \times i = \sqrt{-1} \times \sqrt{-1} = -1\\ i^3 &= i^2 \times i = -1 \times i = -i\\ i^4 &= i^2 \times i^2 = -1 \times -1 = 1\\ i^5 &= i^4 \times i = 1 \times i = i\end{aligned}$$

This pattern now continues and is shown in the diagram:



This reminds us that every fourth multiple comes full circle.

Example

Simplify i^{34}.

Since $i^4 = 1$ we simplify this to the form $i^{4n} \times i^x$.

$$\begin{aligned}\text{Hence } i^{34} &= i^{32} \times i^2\\ &= (i^4)^8 \times i^2\\ &= 1^8 \times -1\\ &= -1\end{aligned}$$

Example

Simplify $15i^7 \times 3i^{18}$.

$$\begin{aligned}15i^7 \times 3i^{18} &= 45i^{25}\\ &= 45i^{24} \times i\\ &= 45(i^4)^6 \times i\\ &= 45(1)^6 \times i\\ &= 45i\end{aligned}$$

Dividing imaginary numbers

This is done in the same way as multiplication.

Example

Simplify $60i^{27} \div 25i^{18}$.

$$\begin{aligned}60i^{27} \div 25i^{18} &= \frac{60i^{27}}{25i^{18}}\\ &= \frac{12}{5}i^9\\ &= \frac{12}{5}(i^4)^2 i\\ &= \frac{12}{5}(1)^2 i\\ &= \frac{12}{5}i\end{aligned}$$

If the power of i in the numerator is lower than the power of i in the denominator then we need to use the fact that $i^4 = 1$.

Example

Simplify $27i^{20} \div 18i^{25}$.

$$\begin{aligned}27i^{20} \div 18i^{25} &= \frac{27i^{20}}{18i^{25}}\\ &= \frac{3}{2i^5}\\ &= \frac{3i^8}{2i^5} \text{ since } i^8 = (i^4)^2 = 1\\ &= \frac{3}{2}i^3\\ &= -\frac{3}{2}i\end{aligned}$$

Hence when performing these operations the answers should not involve powers of i.

Exercise 1

1 Add the following imaginary numbers.
 a $3i + 15i$ b $20i + 18i$
 c $5i + 70i + 35i + 2i$ d $15i + 45i$

2 Subtract the following imaginary numbers.
 a $20i - 8i$
 b $38i - 23i$
 c $56i - 80i$
 d $25i - 31i + 16i - 62i$

3 Multiply the following imaginary numbers giving the answer in the form n or ni where $n \in \mathbb{R}$.
 a $16 \times 15i$ b $4i \times 8i$
 c $15i^2 \times 3i^3$ d $8i \times 12i^4$
 e $9i^2 \times 8i^5$ f $7i^7 \times 5i^5$
 g $3i^2 \times 5i^4 \times 6i^5$

4 Divide the following imaginary numbers giving the answer in the form n or ni where $n \in \mathbb{R}$.
 a $15i^3 \div 2i$ b $6i^7 \div 3i^3$
 c $\frac{15i^3}{6i^2}$ d $16 \div i$

5 Find x if:
 a $xi + 3i^3 - 4i^5 = 2i$ b $\frac{3 + 2i^2}{i} = xi$

6 Simplify these.
 a $3i^3 + 6i^5 - 8i^7 - 2i^9$ b $\frac{2i^3 + 3i^3 - 7i^4}{3i}$
 c $\frac{3i^4}{2i} + \frac{2i^5}{i^2} - \frac{3i}{i^2}$ d $3i^5 \times \frac{2i^6}{6i^3}$
 e $\frac{6i - 3i^2 + 2i^3}{4i}$

17.2 Complex numbers

A **complex number** is defined as one that has a real and an imaginary part. Examples of these would be $2 + 3i$ or $6 - 5i$.

They are generally written in the form $z = x + iy$ where x and y can have any real value including zero.

Hence 6 is a complex number since it can be written in the form $6 + 0i$ and $5i$ is a complex number since it can be written as $0 + 5i$.

Hence both real numbers and imaginary numbers are actually subsets of complex numbers and the notation for this set is $\mathbb{C}$.

Thus we can say $3 + 5i \in \mathbb{C}$.

Adding and subtracting complex numbers

This is done by adding or subtracting the real parts and the imaginary parts in separate groups.

Example

Simplify $(5 + 7i) + (2 - 3i)$.

$$\begin{aligned}(5 + 7i) + (2 - 3i) &= (5 + 2) + (7i - 3i)\\ &= 7 + 4i\end{aligned}$$

Example

Simplify $(9 - 2i) - (4 - 7i)$.

$$\begin{aligned}(9 - 2i) - (4 - 7i) &= (9 - 4) + (-2i - -7i)\\ &= 5 + 5i\end{aligned}$$

Multiplication of complex numbers

This is done by applying the distributive law to two brackets and remembering that $i^2 = -1$. It is similar to expanding two brackets to form a quadratic expression.

Example

Simplify $(2i + 3)(3i - 2)$.

$$\begin{aligned}(2i + 3)(3i - 2) &= 6i^2 + 9i - 4i - 6\\ &= 6(-1) + 9i - 4i - 6\\ &= -12 + 5i\end{aligned}$$

Example

Simplify $(6 + i)(6 - i)$.

$$\begin{aligned}(6 + i)(6 - i) &= 36 + 6i - 6i - i^2\\ &= 36 - (-1)\\ &= 37\end{aligned}$$

We can also use the binomial theorem to simplify complex numbers.

Example

Express $(3 - 2i)^5$ in the form $x + iy$.

$$\begin{aligned}(3 - 2i)^5 &= {}^5C_0(3)^5(-2i)^0 + {}^5C_1(3)^4(-2i)^1 + {}^5C_2(3)^3(-2i)^2\\ &\quad + {}^5C_3(3)^2(-2i)^3 + {}^5C_4(3)^1(-2i)^4 + {}^5C_5(3)^0(-2i)^5\\ &= 243 + 405(-2i) + 270(4i^2) + 90(-8i^3) + 15(16i^4) + (-32i^5)\\ &= 243 + 405(-2i) + 270(-4) + 90(8i) + 15(16) + (-32i)\\ &= -597 - 122i\end{aligned}$$

Division of complex numbers

Before we do this, we have to introduce the concept of a conjugate complex number. Any pair of complex numbers of the form $x + iy$ and $x - iy$ are said to be **conjugate** and $x - iy$ is said to be the conjugate of $x + iy$.

If $x + iy$ is denoted by z, then its conjugate $x - iy$ is denoted by $\bar{z}$ or z^*.

Conjugate complex numbers have the property that when multiplied the result is real. This was demonstrated in the example at the top of the page and the result in general is

$$\begin{aligned}(x + iy)(x - iy) &= x^2 + ixy - ixy - i^2y^2 \\ &= x^2 - (-1)y^2 \\ &= x^2 + y^2\end{aligned}$$

Note the similarity to evaluating the difference of two squares.

To divide two complex numbers we use the property that if we multiply the numerator and denominator of a fraction by the same number, then the fraction remains unchanged in size. The aim is to make the denominator real, and hence we multiply numerator and denominator by the conjugate of the denominator. This process is called realizing the denominator. This is very similar to rationalizing the denominator of a fraction involving surds.

Example

Write $\frac{2 + 3i}{2 - i}$ in the form $a + ib$.

$$\begin{aligned}\frac{2 + 3i}{2 - i} &= \frac{(2 + 3i)}{(2 - i)} \times \frac{(2 + i)}{(2 + i)} \\ &= \frac{4 + 6i + 2i + 3i^2}{4 - 2i + 2i - i^2} \\ &= \frac{4 + 8i - 3}{4 - (-1)} \\ &= \frac{1 + 8i}{5} \\ &= \frac{1}{5} + \frac{8}{5}i\end{aligned}$$

Zero complex number

A complex number is only zero if both the real and imaginary parts are zero, i.e. $0 + 0i$.

Equal complex numbers

Complex numbers are only equal if both the real and imaginary parts are separately equal. This allows us to solve equations involving complex numbers.

Example

Solve $x + iy = (3 + i)(2 - 3i)$.

$$\begin{aligned}x + iy &= (3 + i)(2 - 3i) \\ \Rightarrow x + iy &= 6 + 2i - 9i - 3i^2 \\ \Rightarrow x + iy &= 6 - 7i - 3(-1) \\ \Rightarrow x + iy &= 9 - 7i\end{aligned}$$

Equating the real parts of the complex number gives $x = 9$.
Equating the imaginary parts of the complex number gives $y = -7$.

This idea also allows us to find the square root of a complex number.

Example

Find the values of $\sqrt{3 + 4i}$ in the form $a + ib$.

Let $\sqrt{3 + 4i} = a + ib$

$\Rightarrow (\sqrt{3 + 4i})^2 = (a + ib)^2$

$\Rightarrow 3 + 4i = a^2 + 2iab + i^2b^2$

$\Rightarrow 3 + 4i = a^2 - b^2 + 2iab$

This can also be done in a different way that will be dealt with later in the chapter.

We now use the idea of equal complex numbers and equate the real and imaginary parts.

Equating real parts $\Rightarrow a^2 - b^2 = 3$
Equating imaginary parts $\Rightarrow 2ab = 4 \Rightarrow ab = 2$
These equations can be solved simultaneously to find a and b.

If we substitute $b = \frac{2}{a}$ into $a^2 - b^2 = 3$ we find

$$a^2 - \left(\frac{2}{a}\right)^2 = 3$$

$$\Rightarrow a^4 - 3a^2 - 4 = 0$$

$$\Rightarrow (a^2 - 4)(a^2 + 1) = 0$$

Ignoring the imaginary roots

$\Rightarrow a = 2$ or $a = -2$

$\Rightarrow b = 1$ or $b = -1$

Therefore $\sqrt{3 + 4i} = 2 + i$ or $-2 - i$

As with the square root of a real number, there are two answers and one is the negative of the other.

If we had used the imaginary values for a then

$a = i$ or $a = -i$

$\Rightarrow b = \frac{2}{i}$ or $b = -\frac{2}{i}$

It is usually assumed that a and b are real numbers and we ignore imaginary values for a and b, but if we assume they are imaginary the same answers result.

$\Rightarrow b = \frac{2i^4}{i}$ or $b = -\frac{2i^4}{i}$

$\Rightarrow b = 2i^3$ or $b = -2i^3$

$\Rightarrow b = -2i$ or $b = 2i$

So $\sqrt{3 + 4i} = i + i(-2i)$ or $-i - i(2i)$

$= i - 2i^2$ or $-i + 2i^2$

$= 2 + i$ or $-2 - i$ as before

Complex roots of a quadratic equation

In Chapter 2 we referred to the fact that when a quadratic equation has the property $b^2 - 4ac < 0$, then it has no real roots. We can now see that there are two complex conjugate roots.

Example

Solve the equation $x^2 - 2x + 4 = 0$.

Using the quadratic formula $x = \frac{2 \pm \sqrt{4 - 16}}{2}$

$\Rightarrow x = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm \sqrt{-1}\sqrt{12}}{2}$

$\Rightarrow x = \frac{2 \pm 2i\sqrt{3}}{2}$

$\Rightarrow x = 1 + i\sqrt{3}$ or $x = 1 - i\sqrt{3}$

The $\pm$ sign in the formula ensures that the complex roots of quadratic equations are always conjugate.

Example

Form a quadratic equation which has a complex root of $2 + i$.

Since one complex root is $2 + i$ the other root must be its complex conjugate.

Hence the other root is $2 - i$.

Thus the quadratic equation is

$$[x - (2 + i)][x - (2 - i)] = 0$$

$$\Rightarrow x^2 - (2 + i)x - (2 - i)x + (2 + i)(2 - i) = 0$$

$$\Rightarrow x^2 - 4x + (4 + 2i - 2i - i^2) = 0$$

$$\Rightarrow x^2 - 4x + 5 = 0$$

Complex roots of a polynomial equation

We know from Chapter 4 that solving any polynomial equation with real coefficients always involves factoring out the roots. Hence if any polynomial has complex roots these will always occur in conjugate pairs. For a polynomial equation it is possible that some of the roots will be complex and some will be real. However, the number of complex roots is always even. Hence a polynomial of degree five could have:

- five real roots;
- three real roots and two complex roots; or
- one real root and four complex roots.

Having two real roots and three complex roots is not possible.

To find the roots we need to use long division.

Example

Given that $z = 2 + i$ is a solution to the equation $z^3 - 3z^2 + z + 5 = 0$, find the other two roots.

Since one complex root is $2 + i$ another complex root must be its conjugate.

Hence $2 - i$ is a root.

Thus a quadratic factor of the equation is

$$[z - (2 + i)][z - (2 - i)]$$

$$\Rightarrow z^2 - (2 + i)z - (2 - i)z + (2 + i)(2 - i)$$

$$\Rightarrow z^2 - 4z + (4 + 2i - 2i - i^2)$$

$$\Rightarrow z^2 - 4z + 5$$

Using long division:

$$\begin{array}{r} z + 1 \\ (z^2 - 4z + 5)\overline{)z^3 - 3z^2 + z + 5} \\ -\underline{z^3 - 4z^2 + 5z} \\ z^2 - 4z + 5 \\ -\underline{z^2 - 4z + 5} \\ 0 \end{array}$$

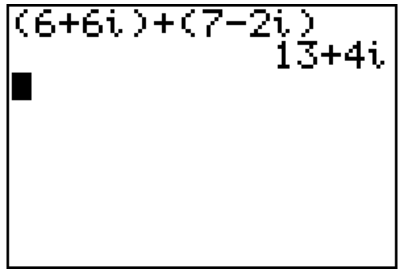
Hence $z^3 - 3z^2 + z + 5 = (z + 1)(z^2 - 4z + 5) = 0$

$\Rightarrow z = -1, 2 + i, 2 - i$

Many of the operations we have covered so far could be done on a calculator.

Example

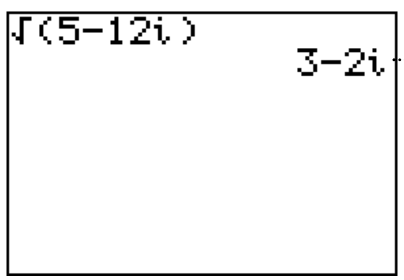
Find $(6 + 6i) + (7 - 2i)$.



$(6 + 6i) + (7 - 2i) = 13 + 4i$

Example

Find $\sqrt{5 - 12i}$.

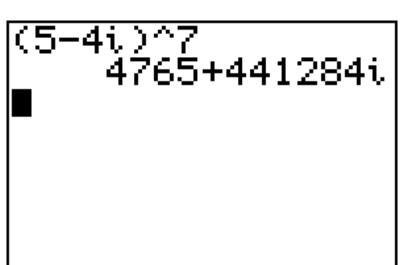


As with real numbers the calculator only gives one value for the square root, unless the negative square root is specified. To find the second square root, we find the negative of the first.

$\sqrt{5 - 12i} = 3 - 2i$ or $-3 + 2i$

Example

Express $(5 - 4i)^7$ in the form $x + iy$.



$(5 - 4i)^7 = 4765 + 441\,284i$

Exercise 2

1 Add these pairs of complex numbers.

a $2 + 7i$ and $6 + 9i$

b $5 + 12i$ and $13 + 16i$

c $4 + 8i$ and $3 - 7i$

d $8 - 7i$ and $6 - 14i$

e $-4 - 18i$ and $-3 + 29i$

2 Find $u - v$.

a $u = 5 + 8i$ and $v = 2 + 13i$

b $u = 16 + 7i$ and $v = 3 - 15i$

c $u = -3 + 6i$ and $v = 17 - 4i$

d $u = -5 - 4i$ and $v = -12 - 17i$

3 Simplify these.

a $(3 + 2i)(2 + 5i)$ **b** $(5 + 3i)(10 + i)$ **c** $i(2 - 7i)$

d $(9 - 5i)(15 - 4i)$ **e** $(15 - 9i)(15 + 9i)$ **f** $(a + bi)(a - bi)$

g $i(5 + 2i)(12 - 5i)$ **h** $(11 + 2i)^2$ **i** $(x + iy)^2$

j $(m + in)^3$ **k** $i(m + 2i)^3(3 - i)$

4 Realize the denominator of each of the following fractions and hence express each in the form $a + bi$.

a $\dfrac{2}{3 - i}$ **b** $\dfrac{5i}{2 - 3i}$ **c** $\dfrac{3 + 4i}{5 - 2i}$ **d** $\dfrac{-5 + 4i}{-2 - 5i}$

e $\dfrac{4 + 12i}{2i}$ **f** $\dfrac{x + iy}{2x - iy}$ **g** $\dfrac{2 + i}{2 - i}$ **h** $i\left(\dfrac{3 + 4i}{3 - i}\right)$

i $\dfrac{2 + 5i}{x - iy}$ **j** $ix\left(\dfrac{3x - iy}{y + ix}\right)$

5 Express these in the form $x + iy$.

a $(1 - i)^4$ **b** $(1 + 2i)^3$ **c** $(3 + 4i)^5$ **d** $(-1 - 3i)^8$

6 Solve these equations for x and y.

a $x + iy = 15 - 7i$ **b** $x + iy = 8$ **c** $x + iy = -3i$

d $x + iy = (3 + 2i)^2$ **e** $x + iy = 6i^2 - 3i$ **f** $x + iy = \dfrac{3i + 2}{i - 4}$

g $x + iy = (2 + 5i)^2$ **h** $(x + iy)^2 = 15i$ **i** $x + iy = \dfrac{2 + 7i}{3 - 4i} - 2$

j $x + iy = 6i^2 - 3i + (2 + i)^2$ **k** $x + iy = \left(\dfrac{2 + i}{3 - 2i}\right)^2 + 15i$

7 Find the real and imaginary parts of these.

a $(6 + 5i)(2 - 3i)$ **b** $\dfrac{3 - 7i}{8 + 11i}$ **c** $\dfrac{3}{2 + 5i} + \dfrac{5}{3 - 4i}$

d $\dfrac{3 - i}{2 + i} + \dfrac{1}{7 - 4i}$ **e** $\dfrac{a}{2 + ib} - \dfrac{3}{4 - ia}$ **f** $\dfrac{x}{x + iy} - \dfrac{x}{x - iy}$

g $(3 + 2i)^5$ **h** $\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)^2$ **i** $[2(1 + i\sqrt{3})]^4$

j $\dfrac{x}{1 + iy} + \dfrac{3}{4 - 3xi}$

8 Find the square roots of these.

a $-15 + 8i$ **b** $1 + i$ **c** $4 - 3i$ **d** $12 + 13i$

e $2 + 5i$ **f** $\frac{2 + i}{3i}$ **g** $\frac{3 - i}{4 + 3i}$ **h** $2i + \frac{3 + 2i}{1 - i}$

i $i\left(\frac{2 - i}{4 + i}\right)^2$

9 Solve these equations.

a $x^2 + 6x + 10 = 0$ **b** $x^2 + x + 1 = 0$ **c** $x^2 + 3x + 15 = 0$

d $3x^2 + 6x + 5 = 0$ **e** $2(x + 4)(x - 1) = 3(x - 7)$

10 Form an equation with these roots.

a $2 + 3i, 2 - 3i$ **b** $3 + i, 3 - i$ **c** $4 + 3i, 4 - 3i$

d $1 + 2i, 1 - 2i, 1$ **e** $3 + 2i, 3 - 2i, 2$ **f** $5 + 2i, 5 - 2i, 3, -3$

11 Find a quadratic equation with the given root.

a $2 + 7i$ **b** $4 - 3i$ **c** $7 - 6i$ **d** $a + ib$

12 Find a quartic equation given that two of its roots are $2 + i$ and $3 - 2i$.

13 Given that $z = 1 + i$ is a root of the equation $z^3 - 5z^2 + 8z - 6 = 0$, find the other two roots.

14 Find, in the form $a + ib$, all the solutions of these equations.

a $z^3 - z^2 - z = 15$ **b** $z^3 + 6z = 20$

15 Given that $z = -2 + 7i$, express $2z + \frac{1}{z}$ in the form $a + ib$, where a and b are real numbers.

16 Given that the complex number z and its conjugate z^* satisfy the equation $zz^* + iz = 66 - 8i$, find the possible values of z.

17 If $z = 12 - 5i$, express $2z + \frac{39}{z}$ in its simplest form.

18 Let z_1 and z_2 be complex numbers. Solve the simultaneous equations

$z_1 + 2z_2 = 4$

$2z_1 + iz_2 = 3 + i$

Give your answer in the form $x + iy$ where $x, y \in \mathbb{Q}$.

19 If $z = 1 + \frac{2}{1 + i\sqrt{3}}$, find z in the form $x + iy$ where $x, y \in \mathbb{R}$.

20 Consider the equation $4(p - iq) = 2q - 3ip - 3(2 + 3i)$ where p and q are real numbers. Find the values of p and q.

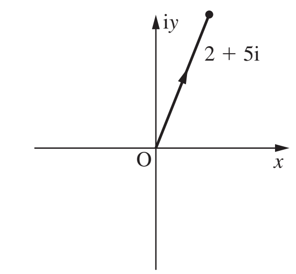
21 If $\sqrt{z} = \frac{3}{1 + 2i} + 4 - 3i$, find z in the form $x + iy$ where $x, y \in \mathbb{R}$.

17.3 Argand diagrams

We now need a way of representing complex numbers in two-dimensional space and this is done on an Argand diagram, named after the mathematician Jean-Robert Argand. It looks like a standard two-dimensional Cartesian plane, except that real numbers are represented on the x-axis and imaginary numbers on the y-axis.

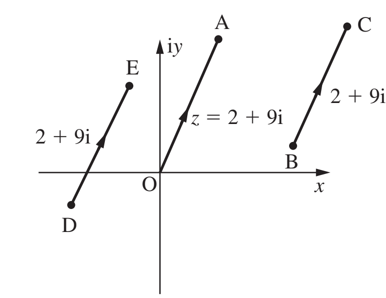
Hence on an Argand diagram the complex number $2 + 5i$ is represented as the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

For this reason it is known as the **Cartesian form** of a complex number.



As with a vector the complex number will usually have an arrow on it to signify the direction and is often denoted by z.

On an Argand diagram the complex number $2 + 9i$ can be represented by the vector $\overrightarrow{OA}$ where A has coordinates (2, 9). However, since it is the line that represents the complex number, the vectors $\overrightarrow{BC}$ and $\overrightarrow{DE}$ also represent the complex number $2 + 9i$.



This is similar to the idea of position vectors and tied vectors introduced in Chapter 12.

Example

The complex numbers $z_1 = \frac{m}{1 - i}$ and $z_2 = \frac{n}{3 + 4i}$, where m and n are real numbers, have the property $z_1 + z_2 = 2$.

a Find the values of m and n.

b Using these values of m and n, find the distance between the points which represent z_1 and z_2 in the Argand diagram.

a

$$\frac{m}{1 - i} + \frac{n}{3 + 4i} = 2$$

$$\Rightarrow \frac{m(1 + i)}{(1 - i)(1 + i)} + \frac{n(3 - 4i)}{(3 + 4i)(3 - 4i)} = 2$$

$$\Rightarrow \frac{m(1 + i)}{2} + \frac{n(3 - 4i)}{25} = 2$$

$$\Rightarrow \frac{m}{2} + \frac{3n}{25} + i\left(\frac{m}{2} - \frac{4n}{25}\right) = 2$$

Equating real parts:

$$\frac{m}{2} + \frac{3n}{25} = 2$$

$\Rightarrow 25m + 6n = 100$ equation (i)

Equating imaginary parts:

$$\frac{m}{2} - \frac{4n}{25} = 0$$

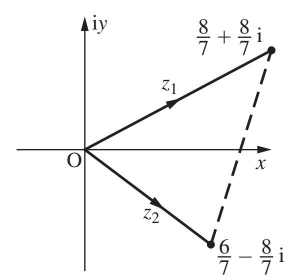
$$\Rightarrow 25m - 8n = 0 \quad \text{equation (ii)}$$

Subtracting equation (ii) from equation (i): $14n = 100 \Rightarrow n = \frac{50}{7}$

Substituting in equation (i): $m = \frac{16}{7}$

b From part **a** $z_1 = \frac{m(1 + i)}{2} = \frac{8(1 + i)}{7}$ and $z_2 = \frac{n(3 - 4i)}{7} = \frac{2(3 - 4i)}{7}$

These are shown on the Argand diagram.

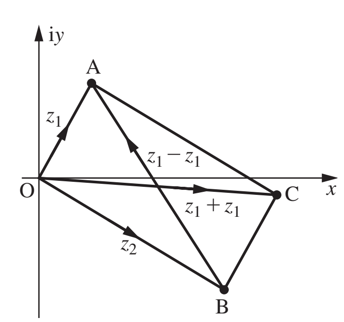


Hence the distance between the points is $\sqrt{\left(\frac{6}{7} - \frac{8}{7}\right)^2 + \left(-\frac{8}{7} - \frac{8}{7}\right)^2} = \frac{\sqrt{260}}{7}$.

Addition and subtraction on the Argand diagram

This is similar to the parallelogram law for vectors which was explained in Chapter 12.

Consider two complex numbers z_1 and z_2 represented by the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



If AC is drawn parallel to OB, then $\overrightarrow{AC}$ also represents z_2. We know from vectors that $\overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$.

Hence $z_1 + z_2$ is represented by the diagonal $\overrightarrow{OC}$.

Similarly $\overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OA}$

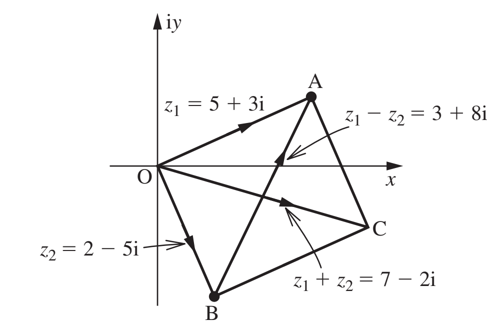
$$\Rightarrow \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$

Hence $z_1 - z_2$ is represented by the diagonal $\overrightarrow{BA}$.

Example

Show $(5 + 3i) + (2 - 5i)$ and $(5 + 3i) - (2 - 5i)$ on an Argand diagram.

Let $z_1 = 5 + 3i$ and $z_2 = 2 - 5i$ and represent them on the diagram by the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Let C be the point which makes OABC a parallelogram.



From the diagram it is clear that $\overrightarrow{OC} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$ and $\overrightarrow{BA} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$ and these two diagonals represent $z_1 + z_2$ and $z_1 - z_2$ respectively. This is confirmed by the fact that $z_1 + z_2 = (5 + 3i) + (2 - 5i) = 7 - 2i$ and $z_1 - z_2 = (5 + 3i) - (2 - 5i) = 3 + 8i$.

Multiplication by i on the Argand diagram

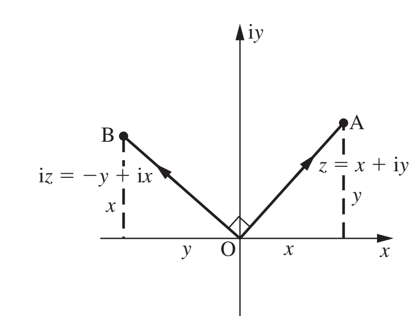
Consider the complex number $z = x + iy$.

Hence

$$iz = ix + i^2y$$

$$\Rightarrow iz = -y + ix$$

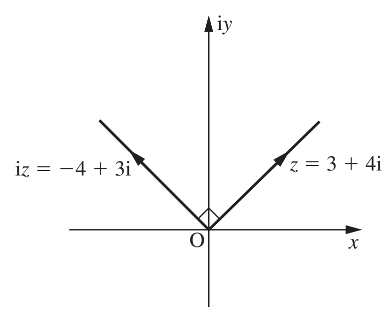
These are shown on the Argand diagram.



Considering this diagram the gradient of OA is $\frac{y}{x}$ and the gradient of OB is $-\frac{x}{y}$. Since the product of gradients is -1, these two lines are perpendicular. Hence if we multiply a complex number by i, the effect on the Argand diagram is to rotate the vector representing it by 90° anticlockwise.

Example

If $z = 3 + 4i$, draw z and iz on an Argand diagram and state iz in the form $a + ib$.



$iz = 3i + 4i^2$

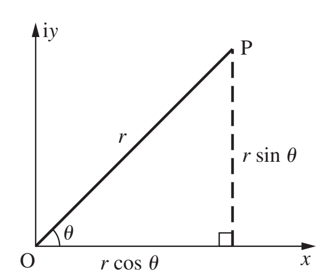
$= -4 + 3i$

Notation for complex numbers

So far we have only seen the representation of a complex number in Cartesian form, that is $x + iy$. However, there are two other forms which are very important.

Polar coordinate form

This is more commonly called the **modulus-argument form** or the mod-arg form. It defines the complex number by a distance r from a given point and an angle θ radians from a given line. Consider the diagram below.



$\overrightarrow{OP}$ represents the complex number $x + iy$. $\overrightarrow{OP}$ has magnitude r and is inclined at an angle of θ radians.

From the diagram $x = r\cos\theta$ and $y = r\sin\theta$.

Thus

$$x + iy = r(\cos\theta + i\sin\theta).$$

This is the modulus-argument form of a complex number where the modulus is r and the angle, known as the argument, is θ. It is usual to give θ in radians. We sometimes express this as (r, θ).

If we are asked to express a complex number given in modulus-argument form in Cartesian form, then we use the fact that $x = r\cos\theta$ and $y = r\sin\theta$.

Example

Express the complex number $\left(2, \frac{\pi}{6}\right)$ in Cartesian form.

$x = r\cos\theta$

$\Rightarrow x = 2\cos\frac{\pi}{6} = \sqrt{3}$

$y = r\sin\theta$

$\Rightarrow y = 2\sin\frac{\pi}{6} = 1$

Hence in Cartesian form the complex number is $\sqrt{3} + i$.

If we are asked to express a complex number given in Cartesian form in modulus-argument form, then we proceed as follows.

If $x = r\cos\theta$ and $y = r\sin\theta$

Then

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$\cos^2\theta + \sin^2\theta = 1 \Rightarrow r = \sqrt{x^2 + y^2}$

The modulus of a complex number is assumed positive and hence we can ignore the negative square root.

Also $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

$\Rightarrow \tan\theta = \frac{y}{x}$

$\Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$

Example

Express $3 + 4i$ in polar form.

$r = \sqrt{x^2 + y^2}$

$\Rightarrow r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$

$\Rightarrow \theta = \arctan\left(\frac{4}{3}\right)$

$\Rightarrow \theta = 0.927\ldots$

Hence in polar form the complex number is $5(\cos 0.927 + i\sin 0.927)$.

This leads us on to the problem of which quadrant the complex number lies in. From the work done in Chapter 1 we know that $\theta = \arctan\left(\frac{y}{x}\right)$ has infinite solutions. To resolve this problem, when calculating the argument in questions like this, it is essential to draw a sketch. Also, by convention, the argument always lies in the range $-\pi < x \leq \pi$. This is slightly different to the method used in Chapter 1 for finding angles in a given range. We will demonstrate this in the example below.

Example

Express the following in modulus-argument form.

a $12 - 5i$

b $-12 + 5i$

c $-12 - 5i$

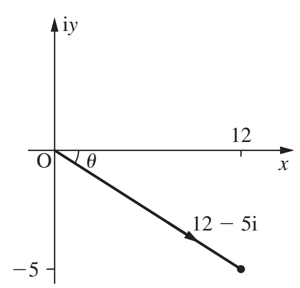
In all cases the modulus is the same since the negative signs do not have an effect.

$$r = \sqrt{x^2 + y^2}$$

$$\Rightarrow r = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$$

In terms of the argument we will examine each case in turn.

a



From the diagram it is clear that the complex number lies in the fourth quadrant and hence the argument must be a negative acute angle.

$$\theta = \arctan\left(\frac{y}{x}\right)$$

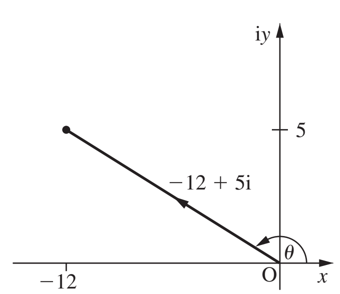
$$\Rightarrow \theta = \arctan\left(\frac{12}{-5}\right)$$

$$\Rightarrow \theta = -1.17\ldots$$

This comes directly from a calculator.

Hence in modulus-argument form the complex number is $13[\cos(-1.18) + i\sin(-1.18)]$.

b



From the diagram it is clear that the complex number lies in the second quadrant and hence the argument must be a positive obtuse angle.

$$\theta = \arctan\left(\frac{y}{x}\right)$$

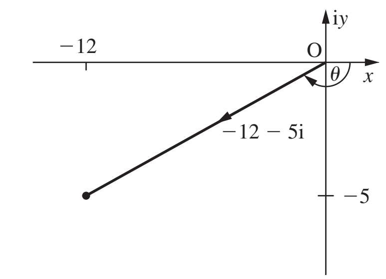
$$\Rightarrow \theta = \arctan\left(\frac{-12}{5}\right)$$

$$\Rightarrow \theta = -1.17\ldots$$

However, this is clearly in the wrong quadrant and hence to find the required angle we need to add π to this giving $\theta = 1.96\ldots$

Hence in modulus-argument form the complex number is $13[\cos(1.97) + i\sin(1.97)]$.

c



From the diagram it is clear that the complex number lies in the third quadrant and hence the argument must be a negative obtuse angle.

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\Rightarrow \theta = \arctan\left(\frac{-12}{-5}\right)$$

$$\Rightarrow \theta = 1.17\ldots$$

However, this is clearly in the wrong quadrant and hence to find the required angle we need to subtract π from this giving $\theta = -1.96\ldots$

Hence in modulus-argument form the complex number is $13[\cos(-1.97) + i\sin(-1.97)]$.

Example

Express $\frac{3}{1 + 2i}$ in polar form.

To do this we begin by expressing the complex number in Cartesian form, by realizing the denominator.

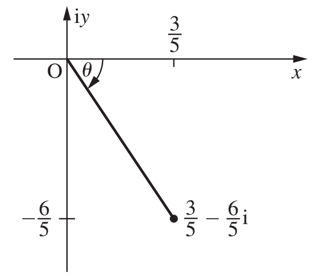
$$\frac{3}{1+2i} = \frac{3(1-2i)}{(1+2i)(1-2i)}$$

$$= \frac{3-6i}{5}$$

Hence $r\cos\theta = \frac{3}{5}$ and $r\sin\theta = -\frac{6}{5}$

$$\Rightarrow r^2 = \left(\frac{3}{5}\right)^2 + \left(-\frac{6}{5}\right)^2$$

$$\Rightarrow r = \sqrt{\frac{9}{5}} = 1.34\ldots$$



From the diagram the complex number lies in the fourth quadrant and hence the argument is a negative acute angle.

$$\frac{r \sin \theta}{r \cos \theta} = \frac{\frac{-6}{5}}{\frac{3}{5}}$$

$$\Rightarrow \tan \theta = -2$$

$$\Rightarrow \theta = -1.10\ldots$$

Hence $\frac{3}{1+2i} = 1.34[\cos(-1.11) + i \sin(-1.11)]$.

Exponential form

This is similar to the mod-arg form and is sometimes called the Euler form. A complex number in this form is expressed as $re^{i\theta}$ where r is the modulus and θ is the argument.

Hence $5\left(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}\right)$ becomes $5e^{i\frac{4\pi}{3}}$ in exponential form.

A calculator will also give complex numbers in exponential form if required.

To express Cartesian form in exponential form or vice versa, we proceed in exactly the same way as changing between polar form and Cartesian form.

We will now show that polar form and exponential form are equivalent.

Let $z = r(\cos \theta + i \sin \theta)$

$$\Rightarrow \frac{dz}{d\theta} = r(-\sin \theta + i \cos \theta)$$

$$\Rightarrow \frac{dz}{d\theta} = ir(\cos \theta + i \sin \theta)$$

$$\Rightarrow \frac{dz}{d\theta} = iz$$

We now treat this as a variables separable differential equation.

$$\Rightarrow \frac{1}{z}\frac{dz}{d\theta} = i$$

$$\Rightarrow \int \frac{1}{z}\frac{dz}{d\theta}\,d\theta = \int i\,d\theta$$

$$\Rightarrow \int \frac{1}{z}\,dz = \int i\,d\theta$$

$$\Rightarrow \ln z = i\theta + \ln c$$

When $\theta = 0$, $z = r \Rightarrow \ln r = \ln c$

$$\Rightarrow \ln z - \ln r = i\theta$$

$$\Rightarrow \ln \frac{z}{r} = i\theta$$

$$\Rightarrow \frac{z}{r} = e^{i\theta} \Rightarrow z = re^{i\theta}$$

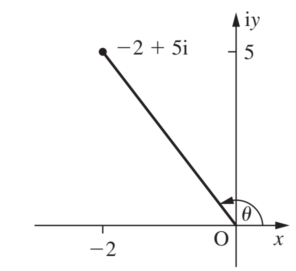
Example

Express $-2 + 5i$ in exponential form.

$r \cos \theta = -2$ and $r \sin \theta = 5$

$$\Rightarrow r^2 = (-2)^2 + (5)^2$$

$$\Rightarrow r = \sqrt{29}$$



From the diagram above the complex number lies in the second quadrant and hence the argument is a positive obtuse angle.

$$\frac{r \sin \theta}{r \cos \theta} = \frac{5}{-2}$$

$$\Rightarrow \tan \theta = -2.5$$

From the calculator $\theta = -1.19\ldots$ However, this is clearly in the wrong quadrant and hence to find the required angle we need to add π to this giving $\theta = 1.95\ldots$

$$\Rightarrow -2 + 5i = \sqrt{29}e^{1.95i}$$

Products and quotients in polar form

If $z_1 = a(\cos \alpha + i \sin \alpha)$ and $z_2 = b(\cos \beta + i \sin \beta)$

Then $z_1z_2 = ab(\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$

$$\Rightarrow z_1z_2 = ab(\cos \alpha \cos \beta + i \sin \alpha \cos \beta + i \cos \alpha \sin \beta + i^2 \sin \alpha \sin \beta)$$

$$\Rightarrow z_1z_2 = ab[(\cos \alpha \cos \beta - \sin \alpha \sin \beta) + i(\sin \alpha \cos \beta + \cos \alpha \sin \beta)]$$

Remembering the compound angle formulae from Chapter 7

$$\Rightarrow z_1z_2 = ab[\cos(\alpha + \beta) + i \sin(\alpha + \beta)]$$

Hence if we multiply two complex numbers in polar form, then we multiply the moduli and add the arguments.

$|z_1z_2| = |z_1| \times |z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$

The standard notation for the modulus of a complex number z is $|z|$ and the standard notation for the argument of a complex number z is arg (z).

Similarly $\frac{z_1}{z_2} = \frac{a}{b}[\cos(\alpha - \beta) + i \sin(\alpha - \beta)]$.

Hence if we divide two complex numbers in polar form, then we divide the moduli and subtract the arguments.

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

Example

Let $z_1 = 2 - \mathrm{i}$ and $z_2 = 3 - \mathrm{i}$.

a Find the product z_1z_2 in the form $x + \mathrm{i}y$.

b Find z_1, z_2 and z_1z_2 in exponential form.

c Hence show that $-\frac{\pi}{4} = \arctan\left(-\frac{1}{2}\right) + \arctan\left(-\frac{1}{3}\right)$.

a $z_1z_2 = (2 - \mathrm{i})(3 - \mathrm{i})$

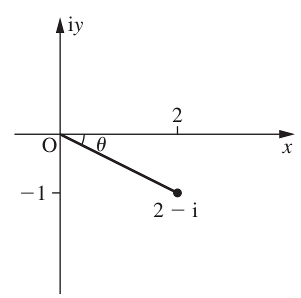
$= 6 - 2\mathrm{i} - 3\mathrm{i} + \mathrm{i}^2$

$= 5 - 5\mathrm{i}$

b For z_1, $r\cos\theta = 2$ and $r\sin\theta = -1$

$\Rightarrow r^2 = (2)^2 + (-1)^2$

$\Rightarrow r = \sqrt{5}$



The diagram shows the complex number lies in the fourth quadrant and hence the argument is a negative acute angle.

$$\frac{r\sin\theta}{r\cos\theta} = \frac{-1}{2}$$

$$\Rightarrow \tan\theta = -\frac{1}{2}$$

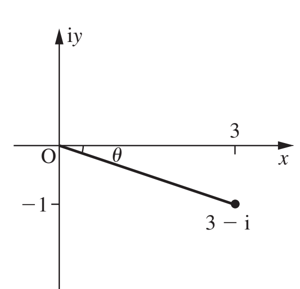
$$\Rightarrow \theta = \arctan\left(-\frac{1}{2}\right)$$

$$\Rightarrow z_1 = \sqrt{5}\mathrm{e}^{\mathrm{i}(\arctan(-\frac{1}{2}))}$$

For z_2, $r\cos\theta = 3$ and $r\sin\theta = -1$

$\Rightarrow r^2 = (3)^2 + (-1)^2$

$\Rightarrow r = \sqrt{10}$



The diagram shows the complex number lies in the fourth quadrant and hence the argument is a negative acute angle.

$$\frac{r\sin\theta}{r\cos\theta} = \frac{-1}{3}$$

$$\Rightarrow \tan\theta = -\frac{1}{3}$$

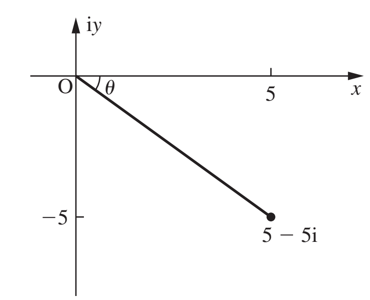
$$\Rightarrow \theta = \arctan\left(-\frac{1}{3}\right)$$

$$\Rightarrow z_2 = \sqrt{10}\mathrm{e}^{\mathrm{i}(\arctan(-\frac{1}{3}))}$$

For z_1z_2, $r\cos\theta = 5$ and $r\sin\theta = -5$

$\Rightarrow r^2 = (5)^2 + (-5)^2$

$\Rightarrow r = 5\sqrt{2}$



The diagram shows the complex number lies in the fourth quadrant and hence the argument is a negative acute angle.

$$\frac{r\sin\theta}{r\cos\theta} = \frac{-5}{5}$$

$$\Rightarrow \tan\theta = -1$$

$$\Rightarrow \theta = -\frac{\pi}{4}$$

$$\Rightarrow z_1z_2 = 5\sqrt{2}\mathrm{e}^{-\mathrm{i}(\frac{\pi}{4})}$$

c Since $\arg(z_1z_2) = \arg z_1 + \arg z_2$

$$-\frac{\pi}{4} = \arctan\left(-\frac{1}{2}\right) + \arctan\left(-\frac{1}{3}\right)$$

Exercise 3

1 If $z_1 = 1 - 2\mathrm{i}$, $z_2 = 2 + 4\mathrm{i}$, $z_3 = -4 + 3\mathrm{i}$ and $z_4 = -5 - \mathrm{i}$, using the parallelogram law, represent these lines on an Argand diagram, showing the direction of each line by an arrow.

a $z_1 + z_3$ **b** $z_2 + z_3$ **c** $z_1 - z_4$ **d** $z_4 + z_1$ **e** $z_3 - z_4$

2 Express these complex numbers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

a $1 + \mathrm{i}\sqrt{3}$ **b** $-2 - 2\mathrm{i}$ **c** $-5 + \mathrm{i}$ **d** $4 - 5\mathrm{i}$ **e** 10 **f** $6\mathrm{i}$

3 Express these complex numbers in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

a $4 + 4\mathrm{i}$ **b** $-3 - 4\mathrm{i}$ **c** $-2 + 7\mathrm{i}$ **d** $1 - 9\mathrm{i}$ **e** 8 **f** $2\mathrm{i}$

4 Express these in the form $a + ib$.

a $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ **b** $\sqrt{5}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

c $10\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ **d** $\sqrt{15}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$

e $3e^{i\frac{3\pi}{4}}$ **f** $\sqrt{5}e^{i\frac{2\pi}{3}}$ **g** $15e^{-i\frac{\pi}{6}}$ **h** $\sqrt{19}e^{-i\frac{3\pi}{8}}$

5 If $m = 5 + 7i$ and $n = 2 - i$, find the modulus and argument of:

a $2m + n$ **b** $3m - 5n$ **c** $2mn$ **d** $\frac{4m}{n}$

6 Find the modulus and argument of each root of these equations.

a $z^2 + 3z + 7 = 0$ **b** $z^2 - 2z + 5 = 0$ **c** $z^2 - 4z + 7 = 0$

7 a Express these complex numbers in exponential form.

i $z_1 = 5 - 12i$ **ii** $z_2 = -3 + 4i$

iii $z_3 = 24 - 7i$ **iv** $z_4 = 1 + i\sqrt{3}$

b Hence find the modulus and argument of:

i z_1z_2 **ii** z_3z_1 **iii** z_4z_1 **iv** $\frac{z_2}{z_4}$ **v** $\frac{z_3}{z_2}$

vi $\frac{z_1}{z_4}$ **vii** $2\frac{z_3}{z_4}$ **viii** $3z_3z_4$

8 a If $z_1 = 3 - 5i$ and $z_2 = 2 - 3i$, draw z_1 and z_2 on an Argand diagram.

b If $z_3 = -iz_1$ and $z_4 = -iz_2$, draw z_3 and z_4 on an Argand diagram.

c Write down the transformation which maps the line segment z_1z_2 onto the line segment z_3z_4.

9 a If $z_1 = 2 - i\sqrt{3}$ and $z_2 = -1 - i$, express z_1 and z_2 in polar form.

b Hence find the modulus and argument of z_1z_2 and $\frac{z_1}{z_2}$.

10 If $z_1 = \frac{3 + i}{2 - 7i}$ and $z_2 = \frac{2 + i}{3 - 2i}$, express z_1 and z_2 in the form $x + iy$.
Sketch an Argand diagram showing the points P representing the complex number $106z_1 + 39z_2$ and Q representing the complex number $106z_1 - 39z_2$.

11 a Show that $z^3 - 1 = (z - 1)(z^2 + z + 1)$.

b Hence find the roots to the equation $z^3 = 1$ in the form $x + iy$.

c Let the two complex roots be denoted by z_1 and z_2. Verify that $z_1 = z_2^2$ and $z_2 = z_1^2$.

12 a The two complex numbers z_1 and z_2 are represented on an Argand diagram. Show that $|z_1 + z_2| \leq |z_1| + |z_2|$.

b If $|z_1| = 3$ and $z_2 = 12 - 5i$, find:

i the greatest possible value of $|z_1 + z_2|$

ii the least possible value of $|z_1 + z_2|$.

13 If $z = \cos\theta + i\sin\theta$ where θ is real, show that $\frac{1}{1 - z} = \frac{1}{2} + i\cot\frac{\theta}{2}$.

14 a Find the solutions to the equation $3z^2 - 4z + 3 = 0$ in modulus-argument form.

b On the Argand diagram, the roots of this equation are represented by the points P and Q. Find the angle POQ.

15 a Find the modulus and argument of the complex number

$$z = \frac{(\sqrt{2} + i)(1 - i\sqrt{2})}{(1 - i)^2}.$$

b Shade the region in the Argand plane such that $\frac{\pi}{2} < \arg\omega < \frac{3\pi}{4}$ and $\frac{1}{2} < |\omega| < 3$ for any complex number ω.

c Determine if z lies in this region.

17.4 de Moivre's theorem

We showed earlier that $z = re^{i\theta}$.

Hence $z^n = (re^{i\theta})^n$

$\Rightarrow z^n = r^n e^{in\theta}$

This is more often stated in polar form.

If $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$.

This is de Moivre's theorem.

An alternative proof of de Moivre's theorem, using the method of proof by induction, will be shown in Chapter 18.

Example

Write $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{15}$ in the form $\cos n\theta + i\sin n\theta$.

Remember: The argument of a complex number lies in the range $-\pi < \theta \leq \pi$.

Using de Moivre's theorem

$$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{15} = \cos\frac{15\pi}{3} + i\sin\frac{15\pi}{3}$$
$$= \cos 5\pi + i\sin 5\pi$$
$$= \cos\pi + i\sin\pi$$

Example

Write $\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^8$ in the form $r(\cos n\theta + i\sin n\theta)$.

$$\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^8 = 2^8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$$
$$= 256\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$$
$$= 256\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Example

Write $\cos\frac{\theta}{5} - i\sin\frac{\theta}{5}$ in the form $(\cos\theta + i\sin\theta)^n$.

We know that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

Hence

$$\cos\frac{\theta}{5} - i\sin\frac{\theta}{5} = \cos\left(-\frac{\theta}{5}\right) + i\sin\left(-\frac{\theta}{5}\right)$$
$$= (\cos\theta + i\sin\theta)^{-\frac{1}{5}}$$

Example

Simplify $\frac{(4\cos 4\theta + 4i\sin 4\theta)(\cos 2\theta - i\sin 2\theta)}{(\cos 3\theta + i\sin 3\theta)}$.

Since de Moivre's theorem is used on expressions of the form $r(\cos n\theta + i\sin n\theta)$ we need to put all expressions in this form:

$$\frac{(4\cos 4\theta + 4i\sin 4\theta)(\cos(-2\theta) + i\sin(-2\theta))}{(\cos 3\theta + i\sin 3\theta)}$$

We now apply de Moivre's theorem:

$$\frac{4(\cos\theta + i\sin\theta)^4(\cos\theta + i\sin\theta)^{-2}}{(\cos\theta + i\sin\theta)^3}$$
$$= 4(\cos\theta + i\sin\theta)^{-1}$$
$$= 4(\cos(-\theta) + i\sin(-\theta))$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$$= 4(\cos\theta - i\sin\theta)$$

Example

Use de Moivre's theorem to derive expressions for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

From de Moivre's theorem we know that

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$$

Using Pascal's triangle or the binomial theorem, we find

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4(\cos^3\theta)(i\sin\theta) + 6(\cos^2\theta)(i\sin\theta)^2$$
$$+ 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$$

Since $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$.

By equating real parts we find $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

And by equating imaginary parts we find $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

This is an alternative way of finding multiple angles of sin and cos in terms of powers of sin and cos.

Example

Using de Moivre's theorem, show that $\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$ where $t = \tan\theta$ and use the equation to solve $t^3 - 3t^2 - 3t + 1 = 0$.

Since we want $\tan 3\theta$ we need expressions for $\sin 3\theta$ and $\cos 3\theta$.
From de Moivre's theorem we know that

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

Using Pascal's triangle we find

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3(\cos^2\theta)(i\sin\theta) + 3(\cos\theta)(i\sin\theta)^2 + (i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

Since $i^2 = -1$ and $i^3 = -i$

By equating real parts we find $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

And by equating imaginary parts we find $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

$$\text{Hence } \tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

$$\Rightarrow \tan 3\theta = \frac{3\frac{\sin\theta}{\cos\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - 3\frac{\sin^2\theta}{\cos^2\theta}}$$

Dividing numerator and denominator by $\cos^3\theta$

$$\Rightarrow \tan 3\theta = \frac{3t - t^3}{1 - 3t^2}$$

Letting $t = \tan\theta$

If we now let $\tan 3\theta = 1$

$$\frac{3t - t^3}{1 - 3t^2} = 1$$
$$\Rightarrow t^3 - 3t^2 - 3t + 1 = 0$$

Hence this equation can be solved using $\tan 3\theta = 1$

$$\Rightarrow 3\theta = -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}$$
$$\Rightarrow \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$$

Hence the solutions to the equation are $\tan\left(-\frac{\pi}{4}\right), \tan\frac{\pi}{12}, \tan\frac{5\pi}{12}$

Example

If $z = \cos\theta + i\sin\theta$, using de Moivre's theorem, show that $\frac{1}{z} = \cos\theta - i\sin\theta$.

$$\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1}$$
$$= \cos(-\theta) + i\sin(-\theta)$$

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$$\frac{1}{z} = \cos\theta - i\sin\theta$$

This now leads to four useful results.

If $z = \cos\theta + i\sin\theta$ and $\frac{1}{z} = \cos\theta - i\sin\theta$, by adding the two equations together we find

$$z+\frac{1}{z}=2\cos\theta$$

If we subtract the two equations we find

$$z-\frac{1}{z}=2\mathrm{i}\sin\theta$$

This can be generalized for any power of z.

If $z=\cos\theta+\mathrm{i}\sin\theta$, then $z^n=(\cos\theta+\mathrm{i}\sin\theta)^n=\cos n\theta+\mathrm{i}\sin n\theta$

and $z^{-n}=(\cos\theta+\mathrm{i}\sin\theta)^{-n}=\cos(-n\theta)+\mathrm{i}\sin(-n\theta)=\cos n\theta-\mathrm{i}\sin n\theta$.

Once again by adding and subtracting the equations we find

$$z^n+\frac{1}{z^n}=2\cos n\theta$$
$$\text{and}\quad z^n-\frac{1}{z^n}=2\mathrm{i}\sin n\theta$$

Example

Using the result $z^n-\frac{1}{z^n}=2\mathrm{i}\sin n\theta$, show that $\sin^3\theta=\frac{3\sin\theta-\sin 3\theta}{4}$.

We know that $z-\frac{1}{z}=2\mathrm{i}\sin\theta$

Hence $\left(z-\frac{1}{z}\right)^3=(2\mathrm{i}\sin\theta)^3=8\mathrm{i}\sin^3\theta$

$\Rightarrow -8\mathrm{i}\sin^3\theta=z^3-3z+\frac{3}{z}-\frac{1}{z^3}$

$=z^3-\frac{1}{z^3}-3\left(z-\frac{1}{z}\right)$

Hence $-8\mathrm{i}\sin^3\theta=2\mathrm{i}\sin 3\theta-6\mathrm{i}\sin\theta$

$\Rightarrow \sin^3\theta=\frac{\sin 3\theta-3\sin\theta}{-4}=\frac{3\sin\theta-\sin 3\theta}{4}$

Roots of complex numbers

Earlier in the chapter we found the square root of a complex number. We can also do this using de Moivre's theorem, which is a much more powerful technique as it will allow us to find any root.

Method

1. Write the complex number in polar form.
2. Add $2n\pi$ to the argument then put it to the necessary power. This will allow us to find multiple solutions.
3. Apply de Moivre's theorem.
4. Work out the required number of roots, ensuring that the arguments lie in the range $-\pi<\theta\leq\pi$. Remember the number of roots is the same as the denominator of the power.

Important points to note

1. The roots are equally spaced around the Argand diagram. Thus for the square root they are π apart. Generally for the nth root they are $\frac{2\pi}{n}$ apart.
2. All the roots have the same moduli.

Example

Find the cube roots of $2+2\mathrm{i}$.

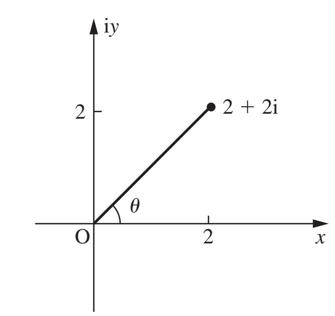
Step 1. Let $2+2\mathrm{i}=r(\cos\theta+\mathrm{i}\sin\theta)$

Equating real and imaginary parts

$\Rightarrow r\cos\theta=2$

$\Rightarrow r\sin\theta=2$

$\Rightarrow r=\sqrt{2^2+2^2}=\sqrt{8}$



The diagram shows the complex number lies in the first quadrant and hence the argument is a positive acute angle.

$\frac{r\sin\theta}{r\cos\theta}=\frac{2}{2}$

$\Rightarrow \tan\theta=1$

$\Rightarrow \theta=\frac{\pi}{4}$

$\Rightarrow 2+2\mathrm{i}=\sqrt{8}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$

Step 2. $(2+2\mathrm{i})^{\frac{1}{3}}=8^{\frac{1}{6}}\left\{\cos\left(\frac{\pi}{4}+2n\pi\right)+\mathrm{i}\sin\left(\frac{\pi}{4}+2n\pi\right)\right\}^{\frac{1}{3}}$

Step 3. $(2+2\mathrm{i})^{\frac{1}{3}}=8^{\frac{1}{6}}\left\{\cos\left(\frac{\pi}{12}+\frac{2n\pi}{3}\right)+\mathrm{i}\sin\left(\frac{\pi}{12}+\frac{2n\pi}{3}\right)\right\}$

Step 4. If we now let $n=-1, 0, 1$ we will find the three solutions.

Hence $(2+2\mathrm{i})^{\frac{1}{3}}=8^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right)$,

$8^{\frac{1}{6}}\left(\cos\left(\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(\frac{3\pi}{4}\right)\right)$, $8^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{12}\right)+\mathrm{i}\sin\left(\frac{\pi}{12}\right)\right)$

These can be converted to the form $x+\mathrm{i}y$.

$(2+2\mathrm{i})^{\frac{1}{3}}=-0.366-1.37\mathrm{i},\ 1.37+0.366\mathrm{i},\ -1+\mathrm{i}$

This calculation can also be done directly on the calculator.

We can also use the exponential form to evaluate roots of a complex number.

Example

Find $(1-\mathrm{i})^{\frac{1}{4}}$.

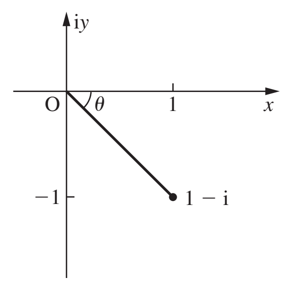
Step 1. Let $1-\mathrm{i} = re^{\mathrm{i}\theta}$

Equating real and imaginary parts

$\Rightarrow r\cos\theta = 1$

$\Rightarrow r\sin\theta = -1$

$\Rightarrow r = \sqrt{1^2+(-1)^2} = \sqrt{2}$



The diagram shows the complex number lies in the fourth quadrant and hence the argument is a negative acute angle.

$\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{1}{-1}$

$\Rightarrow \tan\theta = -1$

$\Rightarrow \theta = -\dfrac{\pi}{4}$

$\Rightarrow 1-\mathrm{i} = \sqrt{2}e^{-\mathrm{i}\frac{\pi}{4}}$

Step 2. $(1-\mathrm{i})^{\frac{1}{4}} = (\sqrt{2}e^{\mathrm{i}(-\frac{\pi}{4}+2n\pi)})^{\frac{1}{4}}$

Step 3. $(1-\mathrm{i})^{\frac{1}{4}} = 2^{\frac{1}{8}}e^{\mathrm{i}(-\frac{\pi}{16}+\frac{n\pi}{2})}$

Step 4. Clearly, if we let $n = -1, 0, 1$ we will find three solutions, but does $n = 2$ or $n = -2$ give the fourth solution? Since $\dfrac{\pi}{16}$ is negative, then using $n = -2$ takes the argument out of the range $-\pi < x \leq \pi$. Hence we use $n = 2$.

Thus $(1-\mathrm{i})^{\frac{1}{4}} = 2^{\frac{1}{8}}e^{-\mathrm{i}\frac{9\pi}{16}}, 2^{\frac{1}{8}}e^{-\mathrm{i}\frac{\pi}{16}}, 2^{\frac{1}{8}}e^{\mathrm{i}\frac{7\pi}{16}}, 2^{\frac{1}{8}}e^{\mathrm{i}\frac{15\pi}{16}}$

These can be converted to the form $x + \mathrm{i}y$.

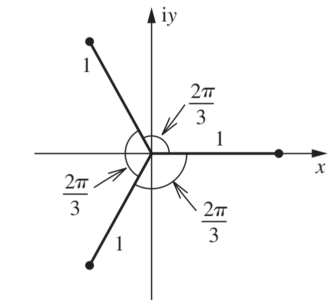
$(1-\mathrm{i})^{\frac{1}{4}} = -0.213 - 1.07\mathrm{i}, 1.07 - 0.213\mathrm{i}, 0.213 + 1.07\mathrm{i}, -1.07 + 0.213\mathrm{i}$

Again, this calculation can also be done directly on the calculator.

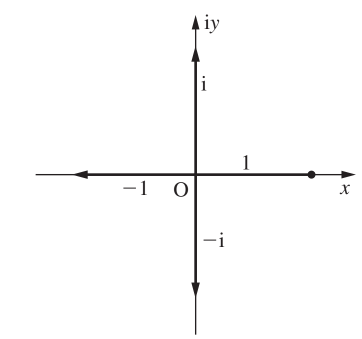
Roots of unity

We can find the complex roots of 1 and these have certain properties.

1. Since the modulus of 1 is 1, then the modulus of all roots of 1 is 1.
2. We know that the roots are equally spaced around an Argand diagram. Since one root of unity will always be 1, we can measure the arguments relative to the real axis.
   Hence the cube roots of unity on an Argand diagram will be:



The fourth roots of unity will be:



3. Since the roots of unity are equally spaced and all have modulus 1, if we call one complex root b, say, then for the cube roots of unity the other roots will be 1 and b^2. Similarly for the fifth roots, if one complex root is b, then the other roots will be $1, b^2, b^3$ and b^4.

Example

a Simplify $(\omega-1)(1+\omega+\omega^2)$.

b Hence factorise $z^3 = 1$.

c If ω is a complex root of this equation, simplify:

i ω^3

ii $1+\omega+\omega^2$

iii ω^4

iv $(\omega-1)(\omega^2+\omega)$

a $(\omega-1)(1+\omega+\omega^2) = \omega+\omega^2+\omega^3-1-\omega-\omega^2$

$= \omega^3 - 1$

b $z^3 = 1 \Rightarrow z^3 - 1 = 0$

$\Rightarrow (z-1)(1+z+z^2) = 0$

c

i Since $z = \omega$, $\omega^3 = 1$

ii Since $z = \omega$ and from part **b** $1+z+z^2 = 0$, $1+\omega+\omega^2 = 0$

iii $\omega^4 = \omega^3 \times \omega$

Since $\omega^3 = 1$, $\omega^4 = 1 \times \omega = \omega$

iv $(\omega-1)(\omega^2+\omega) = \omega^3+\omega^2-\omega^2-\omega$

$= 1 - \omega$

Exercise 4

1 Use de Moivre's theorem to express each of these complex numbers in the form $r(\cos n\theta + i\sin n\theta)$.

a $[2(\cos\theta + i\sin\theta)]^{10}$ **b** $(\cos\theta + i\sin\theta)^{25}$ **c** $[3(\cos\theta + i\sin\theta)]^{-5}$

d $(\cos\theta + i\sin\theta)^{-9}$ **e** $(\cos\theta + i\sin\theta)^{\frac{1}{2}}$ **f** $[4(\cos\theta + i\sin\theta)]^{-\frac{1}{3}}$

g $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^6$ **h** $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^9$ **i** $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{-5}$

j $\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^{\frac{1}{2}}$

2 Express each of these in the form $r(\cos\theta + i\sin\theta)^n$.

a $\cos 7\theta + i\sin 7\theta$ **b** $4\cos\frac{1}{2}\theta + 4i\sin\frac{1}{2}\theta$

c $6\cos(-3\theta) + 6i\sin(-3\theta)$ **d** $\cos\left(-\frac{1}{4}\theta\right) + i\sin\left(-\frac{1}{4}\theta\right)$

e $\cos 2\theta - i\sin 2\theta$ **f** $\cos\frac{1}{8}\theta - i\sin\frac{1}{8}\theta$

3 Simplify these expressions.

a $(\cos 3\theta + i\sin 3\theta)(\cos 5\theta + i\sin 5\theta)$

b $(\cos 2\theta + i\sin 2\theta)\left(\cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta\right)$

c $\dfrac{(\cos 8\theta + i\sin 8\theta)}{(\cos 5\theta + i\sin 5\theta)}$

d $\dfrac{(\cos 4\theta + i\sin 4\theta)}{(\cos 5\theta + i\sin 5\theta)}$

e $\dfrac{(\cos 10\theta + i\sin 10\theta)(\cos 2\theta + i\sin 2\theta)}{(\cos\theta + i\sin\theta)}$

f $(\cos 4\theta + i\sin 4\theta)(\cos 7\theta - i\sin 7\theta)$

g $\left(\cos\frac{1}{3}\theta + i\sin\frac{1}{3}\theta\right)\left(\cos\frac{1}{2}\theta - i\sin\frac{1}{2}\theta\right)$

h $\dfrac{\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^2}{\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^4}$

i $\dfrac{\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)^5\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)^{-2}}{\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)^4}$

j $\sqrt[4]{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$

4 Use de Moivre's theorem to find these roots.

a the square root of $-5 + 12i$ **b** the square root of $-2 - 2i$

c the cube roots of $1 - i$ **d** the cube root of $3 - 5i$

e the fourth roots of $3 + 4i$ **f** the fifth roots of $-5 - 12i$

g the sixth roots of $\sqrt{3} + i$

5 Without first calculating them, illustrate the nth roots of unity on an Argand diagram where n is:

a 3 **b** 6 **c** 8 **d** 9

6 a Express the complex number $16i$ in polar form.

b Find the fourth roots of $16i$ in both polar form and Cartesian form.

7 a Write $1 + i\sqrt{3}$ in polar form.

b Hence find the real and imaginary parts of $(1 + i\sqrt{3})^{16}$.

8 Prove those trigonometric identities using methods based on de Moivre's theorem.

a $\sin 3\theta \equiv 3\cos^2\theta\sin\theta - \sin^3\theta$

b $\tan 6\theta \equiv 2\left(\dfrac{3 - 10t^2 + 3t^4}{1 - 15t^2 + 15t^4 - t^6}\right)$ where $t = \tan\theta$

9 a Use de Moivre's theorem to show that $\tan 4\theta = \dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$ where $t = \tan\theta$.

b Use your result to solve the equation $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$.

10 Let $z_1 = m\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = m\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$. Express $\left(\frac{z_1}{z_2}\right)^4$ in the form $x + iy$.

11 Let $z_1 = r\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = 3 - 4i$.

a Write z_2 in modulus-argument form.

b Find r if $|z_1 z_2^2| = 4$.

12 Given that ω is a complex cube root of unity, $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$, simplify each of the expressions $(1 + 3\omega + \omega^2)$ and $(1 + \omega + 3\omega^2)$ and find the product and the sum of these two expressions.

13 By considering the ninth roots of unity, show that:

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = -\frac{1}{2}$$

14 a If $z = \cos\theta + i\sin\theta$, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ and

$z^n - \dfrac{1}{z^n} = 2i\sin n\theta$.

b Hence show that:

i $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$

ii $\sin^6\theta = \frac{1}{32}(-\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10)$

15 Consider $z^7 = 128$.

a Find the root to this equation in the form $r(\cos\theta + i\sin\theta)$ which has the smallest positive argument. Call this root z_1.

b Find $z_1^2, z_1^3, z_1^4, z_1^5, z_1^6, z_1^7$ in modulus-argument form.

c Plot the points that represent $z_1^2, z_1^3, z_1^4, z_1^5, z_1^6, z_1^7$ on an Argand diagram.

d The point z_1^n is mapped to z_1^{n+1} by a composition of two linear transformations. Describe these transformations.

16 a Show that $-i$ satisfies the equation $z^3 = i$.

b Knowing that the three roots of the equation $z^3 = i$ are equally spaced around the Argand diagram and have equal moduli, write down the other two roots, z_1 and z_2, of the equation in modulus-argument form. (z_1 lies in the second quadrant.)

c Find the complex number ω such that $\omega z_1 = z_2$ and $\omega z_2 = -\text{i}$.

17 The complex number z is defined by $z = \cos\theta + \text{i}\sin\theta$.

a Show that $\frac{1}{z} = \cos(-\theta) + \text{i}\sin(-\theta)$.

b Deduce that $z^n + \frac{1}{z^n} = 2\cos n\theta$.

c Using the binomial theorem, expand $(z + z^{-1})^6$.

d Hence show that $\cos^6\theta = a\cos 6\theta + b\cos 4\theta + c\cos 2\theta + d$ giving the values of a, b, c and d.

Review exercise

1 Find the modulus and argument of the complex number $\frac{5 - 7\text{i}}{1 + 2\text{i}}$.

2 Find the real number k for which $1 + k\text{i}$, $(\text{i} = \sqrt{-1})$, is a zero of the polynomial $z^2 + kz + 5$. [IB Nov 00 P1 Q10]

3 If $z = 1 + 2\text{i}$ is a root of the equation $z^2 + az + b$, find the values of a and b.

4 If z is a complex number and $|z + 16| = 4|z + 1|$, find the value of $|z|$. [IB Nov 00 P1 Q18]

5 **a** Show that $(1 + \text{i})^4 = -4$.

b Hence or otherwise, find $(1 + \text{i})^{64}$.

6 Solve the equation $\frac{-\text{i}}{x - \text{i}y} = \frac{4 + 7\text{i}}{5 - 3\text{i}}$ for x and y, leaving your answers as rational numbers. [IB May 94 P1 Q15]

7 Find a cubic equation with real coefficients, given that two of its roots are 3 and $1 - \text{i}\sqrt{3}$.

8 If $z = x + \text{i}y$, find the real part and the imaginary part of $z + \frac{1}{z}$.

9 Given that $z = (b + \text{i})^2$, where b is real and positive, find the exact value of b when $\arg z = 60°$. [IB May 01 P1 Q14]

10 **a** If $z = 1 + \text{i}\sqrt{3}$, find the modulus and argument of z.

b Hence find the modulus and argument of z^2.

c **i** On an Argand diagram, point A represents the complex number $0 + \text{i}$, B represents the complex number z and C the complex number z^2. Draw these on an Argand diagram.

ii Calculate the area of triangle OBC where O is the origin.

iii Calculate the area of triangle ABC.

11 **a** Verify that $(z - 1)(1 + z + z^2) = z^3 - 1$.

b Hence or otherwise, find the cube roots of unity in the form $a + \text{i}b$.

c Find the cube roots of unity in polar form and draw them on an Argand diagram.

d These three roots form the vertices of a triangle. State the length of each side of the triangle and find the area of the triangle.

12 Given that $(2 - 3\text{i})a + 3b = 2 + 5\text{i}$, find the values of a and b if

a a and b are real

b a and b are conjugate complex numbers.

13 Let $y = \cos\theta + \text{i}\sin\theta$.

a Show that $\frac{dy}{d\theta} = \text{i}y$.

[You may assume that for the purposes of differentiation and integration, i may be treated in the same way as a real constant.]

b Hence show, using integration, that $y = \text{e}^{\text{i}\theta}$.

c Use this result to deduce de Moivre's theorem.

d **i** Given that $\frac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, where $\sin\theta \neq 0$, use de Moivre's theorem with $n = 6$ to find the values of the constants a, b and c.

ii Hence deduce the value of $\lim\limits_{\theta\to 0}\frac{\sin 6\theta}{\sin\theta}$. [IB Nov 06 P2 Q5]

14 Given that z and ω are complex numbers, solve the simultaneous equations

$z + \omega = 11$

$\text{i}z + 5\omega = 29$

expressing your solution in the form $a + b\text{i}$ where a and b are real. [IB Nov 89 P1 Q20]

15 Let $z = \cos\theta + \text{i}\sin\theta$ for $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

a **i** Find z^3 using the binomial theorem.

ii Use de Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

b Hence prove that $\frac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} = \tan\theta$.

c Given that $\sin\theta = \frac{1}{3}$, find the exact value of $\tan 3\theta$. [IB May 06 P2 Q2]

16 Consider the complex number $z = \frac{\left(\cos\frac{\pi}{4} - \text{i}\sin\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{24} - \text{i}\sin\frac{\pi}{24}\right)^4}$.

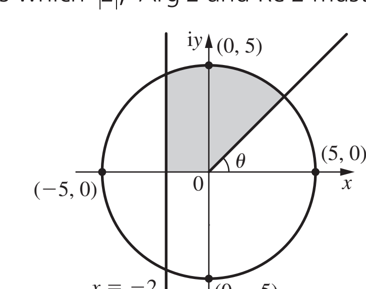
a **i** Find the modulus of z.

ii Find the argument of z, giving your answer in radians.

b Using de Moivre's theorem, show that z is a cube root of one, i.e. $z = \sqrt[3]{1}$.

c Simplify $(1 + 2z)(2 + z^2)$, expressing your answer in the form $a + b\text{i}$, where a and b are exact real numbers. [IB Nov 02 P2 Q2]

17 In this Argand diagram, a circle has centre the origin and radius 5, $\theta = \frac{\pi}{3}$ and the line which is parallel to the imaginary axis has equation $x = -2$. The complex number z corresponds to a point inside, or on, the boundary of the shaded region. Write down inequalities which $|z|$, Arg z and Re z must satisfy. (Re z means the real part of z.)



18 Let $z = 3 + ik$ and $\omega = k + 7i$ where $k \in \mathbb{R}$ and $i = \sqrt{-1}$.

a Express $\frac{z}{\omega}$ in the form $a + ib$ where $a, b \in \mathbb{R}$.

b For what values of k is $\frac{z}{\omega}$ a real number?

19 **a** Find all three solutions of the equation $z^3 = 1$ where z is a complex number.

b If $z = \omega$ is the solution of the equation $z^3 = 1$ which has the smallest positive root, show that $1 + \omega + \omega^2 = 0$.

c Find the matrix product $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$ giving your answer in its simplest form (that is, not in terms of ω).

d Solve the system of simultaneous equations

$x + y + z = 3$

$x + \omega y + \omega^2 z = -3$

$x + \omega^2 y + \omega z = -3$

giving your answer in numerical form (that is, not in terms of ω). [IB Nov 98 P2 Q4]

20 z_1 and z_2 are complex numbers on the Argand diagram relative to the origin. If $|z_1 + z_2| = |z_1 - z_2|$, show that $\arg z_1$ and $\arg z_2$ differ by $\frac{\pi}{2}$.

21 **a** Find the two square roots of $3 - 4i$ in the form $x + iy$ where x and y are real.

b Draw these on the Argand diagram, labelling the points A and B.

c Find the two possible points C_1 and C_2 such that triangles ABC_1 and ABC_2 are equilateral.

18 Mathematical Induction

Abu Bekr ibn Muhammad ibn al-Husayn Al-Karaji was born on 13 April 953 in Baghdad, Iraq and died in about 1029. His importance in the field of mathematics is debated by historians and mathematicians. Some consider that he only reworked previous ideas, while others see him as the first person to use arithmetic style operations with algebra as opposed to geometrical operations.

In his work, *Al-Fakhri*, Al-Karaji succeeded in defining $x, x^2, x^3, \ldots$ and $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \ldots$ and gave rules for finding the products of any pair without reference to geometry. He was close to giving the rule $x^n x^m = x^{m+n}$ for all integers n and m but just failed because he did not define $x^0 = 1$.

In his discussion and demonstration of this work Al-Karaji used a form of mathematical induction where he proved a result using the previous result and noted that this process could continue indefinitely. As we will see in this chapter, this is not a full proof by induction, but it does highlight one of the major principles.

Al-Karaji used this form of induction in his work on the binomial theorem, binomial coefficients and Pascal's triangle. The table shown is one that Al-Karaji used, and is actually Pascal's triangle in its side.

He also worked on the sums of the first n natural numbers, the squares of the first n natural numbers and the cubes of these numbers, which we introduced in Chapter 6.

col 1	col 2	col 3	col 4	col 5	...
1	1	1	1	1	...
1	2	3	4	5	...
	1	3	6	10	...
		1	4	10	...
			1	5	...
				1	...